Problem Set 3 (10 points)

Consider the following hypothetical situation in World War II. During the invasion of the Solomon Islands by the United States, there was a period when U.S. forces had occupied some of the southern islands but not yet fully captured the northwestern island of Bougainville. A U.S. admiral had to maintain supply to Bougainville in the face of Japanese defensive operations. U.S. ships had two routes that they could take for resupply: a northern route and a southern route around the intervening islands. Every night when the supply run was made, one of these routes had to be picked. The northern route was a little bit quicker, and so, all other things being equal, the Americans would prefer to go north. The Japanese had enough ships to block one route, but not both (if they tried to block both, their force would be outnumbered and the Americans would get through). The Japanese had to guess which route the Americans would be using. The payoffs are as follows:

- If the two forces meet up, whether north or south, the U.S. supply ships will be sunk, the outcome the United States least prefers. The Japanese prefer this outcome. The Japanese payoff here is 200; the U.S. payoff is -100.
- If the United States goes south while the Japanese search north, the supplies arrive slowly. The United States gets a payoff of 50 for this, while the Japanese receive -100.
- If the United States goes north while the Japanese search south, the supplies arrive quickly. The United States gets a payoff of 125 for this, while the Japanese receive -100.

The preference ordering is as follows: If S designates south, N designates north, then:
United States: $NS \succ SN \succ NN \sim SS$
Japan: $NN \sim SS \succ NS \sim SN$

a. Draw the game matrix with payoffs. (1 point)

b. Look for pure strategy Nash equilibria in this game. Is there a stable point at which we can expect the actors to settle down? If so, what is it? (1 point)

c. What would happen to the United States if it picked a single strategy (N or S)? (1 point)

d. What would happen to Japan if it picked a single strategy (N or S)? (1 point)

Let us suppose the players want to mix their strategies, creating a mixed-strategy equilibrium. A mixed strategy works by making the opponent indifferent between his or her choices. To do this, we need to calculate the probability that we will take each different action so as to make the opponent's expected payoff from each of its actions equal. To put it specifically in this case, the United States wants to pick a probability of going north that will make $EU_{\text{defend north}} = EU_{\text{defend south}}$ for the Japanese. This makes the Japanese indifferent between defending north and south. This will negate any Japanese advantage from picking a better strategy.

We must solve for the probability P_{north} that the United States must adopt to make the Japanese indifferent. To start, the Japanese expected payoff equations are:

$$EU_{\text{defend north}} = P_{north} * 200 + (1 - P_{north}) * (-100)$$
$$EU_{\text{defend south}} = P_{north} * (-100) + (1 - P_{north}) * (200)$$

We can then solve for the value of P_{north} that makes these equal:

$$EU_{\text{defend north}} = EU_{\text{defend south}} \quad \Rightarrow$$
$$P_{north} * 200 + (1 - P_{north}) * (-100) = P_{north} * (-100) + (1 - P_{north}) * (200) \quad \Rightarrow$$
$$P_{north} = 0.5$$

If the United States sails north with probability 0.5, then the Japanese will be indifferent between defending north and south. In this case, the United States will prevent any advantage to the Japanese going north or south. Be sure you understand why the Japanese expected payoff equations are as listed.

e. We said that the Japanese are indifferent between going north and south. This means that the expected payoff from defending each route is identical. Given that the United States has a 0.5 probability of sailing north, and using the expected payoff equations above, compute the expected payoff to Japan of defending north and south. Show your work. (2 points)

The Japanese also might want to make sure that the United States is not gaining any advantage. They can do this by using a mixed-strategy equilibrium to keep the United States indifferent between its options. The Japanese want to defend the northern route with probability q_{north}. Compute the mixed-strategy equilibrium probability for defending north.

f. Write out the U.S.'s expected payoff from sailing north, given that Japan is defending north with probability q_{north}. (1 point)

g. Write out the U.S.'s expected payoff from sailing south, given that Japan is defending north with probability q_{north}. (1 point)

h. Solve for the value of q_{north} that makes the U.S. indifferent between sailing north and south. Show your work. (1 point)

i. If your calculation is correct and Japan is defending north with the probability of q_{north}, then the expected payoff to the United States of sailing north and south should be the same. Compute $EU_{\text{sail north}}$ and $EU_{\text{sail south}}$. (1 point)